Problem involving Mobius function in $\mathbb{C}$.
https://www.linkedin.com/feed/update/urn:li:activity:6520090253649416192
This continued complex number fraction is periodical and its values are $(1-i)/2, 2/(1+i), 1, 1/(1+i)$. Do you another complex number continued fraction with similar property ?

$$f_n(z) = \frac{1}{1+f_{n-1}(z)}$$

if $f_n(z)$ does not converge then $\frac{1}{1+f_{n-1}(z)}$ will be periodic

Example $z_{n+1} = f(z_n)$, where $f(z) := \frac{1}{i+z}$ create periodic sequence if $z_1 := \frac{1}{1+i}$ then $z_2 = f(z_1) = \frac{1}{i+\frac{1}{1+i}} = 1-i$, $z_3 = f(z_2) = \frac{1}{i+1-i} = 1, z_4 = f(z_3) = \frac{1}{i+1} = z_1$.

$$z_{n+1} = \frac{1}{i+z_n} \Leftrightarrow iz_{n+1} = \frac{i^2}{i(i+z_n)} \Leftrightarrow iz_{n+1} = -\frac{1}{iz_n-1} \Leftrightarrow iz_{n+1}-1 = -\frac{1}{iz_n-1}-1.$$

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One question was related with the sequence (z_n) defined by $z_n = h(z_{n-1}), n \in \mathbb{N}$, where $h(z) := \frac{1}{1+z}$ and $z_0 = z \in \mathbb{C}$.

Let $h_{n+1} := h \circ h_n, n \in \mathbb{N}$,where $h_1 = h$ and let $H_n := \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ is matrix of Mobius function $h_n(z) = \frac{h_{11}z+h_{12}}{h_{21}z+h_{22}}$ *. In particular, $H_1 = H = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is matrix of $h(z)$

Noting that $H_n = H^n$ and $H = \begin{pmatrix} f_0 & f_1 \\ f_1 & f_2 \end{pmatrix}$, where f_n is n-th Fibonacci number defined by $f_{n+1} = f_n + f_{n-1}, n \in \mathbb{N}$ with initial conditions $f_0 = 0, f_1 = 1$, we obtain

$$H^2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_3 \end{pmatrix}, H^3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} f_2 & f_3 \\ f_3 & f_4 \end{pmatrix}$$

Supposition $H^n = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix}$ implies $H^{n+1} = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} =$

$$\begin{pmatrix} f_n & f_n+f_{n-1} \\ f_{n+1} & f_n+f_{n+1} \end{pmatrix} = \begin{pmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{pmatrix}.$$

Thus, by Math Induction prove that

$h_n(z) = \frac{zf_{n-1}+f_n}{zf_n+f_{n+1}}$ and $h_n(z) = z \Leftrightarrow \frac{zf_{n-1}+f_n}{zf_n+f_{n+1}} = z \Leftrightarrow zf_{n-1}+f_n = z^2f_n+zf_{n+1} \Leftrightarrow$
$z^2f_n + z(f_{n+1}-f_{n-1}) - f_n = 0 \Leftrightarrow z^2f_n + zf_n - f_n = 0 \Leftrightarrow z^2+z-1 \Leftrightarrow h(z) = z \Leftrightarrow$
$z = \frac{\sqrt{5}-1}{2}$ or $z = \frac{-\sqrt{5}-1}{2}$.

So, $h(z)$ has n-fixed point only if $n = 1$ and these points are $z = \frac{\sqrt{5}-1}{2}$ or

$z = \frac{-\sqrt{5}-1}{2}$.

that is the sequence (z_n), where $z_n = \frac{1}{1+z_{n-1}}, n \in \mathbb{N}$ is a constant sequence if $z_0 \in \left\{\frac{\sqrt{5}-1}{2}, \frac{-\sqrt{5}-1}{2}\right\}$ and is non periodic if $z_0 \notin \left\{\frac{\sqrt{5}-1}{2}, \frac{-\sqrt{5}-1}{2}\right\}$.

Another question was related to the sequence (z_n) defined by $z_{n+1} = h(z_n), n \in \mathbb{N}$ where $h(z) := \frac{1}{i+z}$, $z_1 = z \in \mathbb{C}$

and was accompanied with example of 3-periodic sequence

$\frac{1-i}{2}, \frac{2}{1+i}, 1, \frac{1}{1+i}, \ldots$,(here $\frac{1}{1+i} = \frac{1-i}{2}$).

In fact the sequence (z_n) defined by $z_{n+1} = \frac{1}{i+z_n}, n \in \mathbb{N}$ and $z_1 = z$ is 3-periodic for any $z \in \mathbb{C} \setminus \{-i, 0\}$.

Indeed, since $h_2(z) = \frac{1}{i+\frac{1}{i+z}} = \frac{1-iz}{z}, h_3(z) = \frac{1}{i+\frac{1-iz}{z}} = z$ then any $z \in \mathbb{C} \setminus \{-i, 0\}$ is fixed point of $h_3(z)$, that is $h_3(z) = z$ for any $z \in \mathbb{C} \setminus \{-i, 0\}$. Or by the other words sequence (z_n) is 3-periodic.

* For any Mobius function $h_n(z) = \frac{h_{11}z + h_{12}}{h_{21}z + h_{22}}$ if $H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ is matrix of $h_n(z)$ then for any $c \in \mathbb{C} \setminus \{0\}$ matrix $cH = \begin{pmatrix} ch_{11} & ch_{12} \\ ch_{21} & ch_{22} \end{pmatrix}$ is matrix of $h_n(z)$ as well.